Content of these notes

1 Model ...................................................................................................................................................................... 1
2 Derivations.......................................................................................................................................................... 2
   2.1 Limit equilibrium equations .......................................................................................................................... 2
   2.2 The vertical loading V ................................................................................................................................. 2
   2.3 The shear force Ts acting at the lateral slip surfaces .................................................................................... 3
   2.4 Special case: Short-term stability in a low permeability ground ............................................................... 4
3 Summary of equations ....................................................................................................................................... 5

1 Model

Based upon the slip surface observed in model tests (Fig. 1), a failure mechanism will be considered consisting of a wedge in front of the face and a prismatic body extending up to the surface (Fig. 2). Fig. 3 shows the geometrical parameters of the problem. Note that the angle $\omega$ will be calculated iteratively (criterion: maximization of support force $S$).

As the deformation of the ground is not taken into account in a limit equilibrium analysis, only the shear strength parameters are considered, i.e. the cohesion $c$ and the friction angle $\phi$ (homogeneous and isotropic ground obeying Coulomb failure criterion).

Fig. 4 shows the forces acting upon the wedge: the weight $G$, the vertical load $V$ resulting from the prismatic body, the shear and the normal force $(T, N)$ at the inclined slip surface, the shear and normal forces $(T_s, N_s)$ at the lateral slip surfaces and the face support force $S$. For the sake of simplicity, the horizontal shear force acting at the upper boundary of the wedge will be neglected.

Figure 1. Development of slip surfaces in a model test.
**Fig. 2:** Failure mechanism consisting of a wedge and a prismatic body.

**Fig. 3:** Geometrical parameters.

**Fig. 4:** Forces
2 Derivations

2.1 Limit equilibrium equations

The equilibrium equation in the direction of sliding is:

\[(V + G) \cos \omega = T + T_s + S \sin \omega,\]  

(1)

The equilibrium equation in the direction of the normal force N is:

\[N = S \cos \omega + (V + G) \sin \omega.\]  

(2)

The shear force T fulfills Coulomb criterion:

\[T = N \tan \phi + c B H \cos \omega.\]  

(3)

Elimination of the forces N und T leads to the following equation for the face support force S:

\[S = \frac{V + G}{\tan(\omega + \phi)} - \frac{T_s + c B H}{\cos \omega (\tan \omega + \tan \phi)}.\]  

(4)

The weight of the wedge is given by:

\[G = \frac{1}{2} B H^2 \tan \omega \quad (\gamma = \text{unit weight of the ground}).\]  

(5)

In the next sections, expressions will be derived for the forces V and T_s appearing on the right hand side of Eq. (4).

2.2 The vertical loading V

The vertical force V is calculated by applying silo-theory on the prismatic body:

\[V = F \sigma_v,\]  

(6)

where F und \(\sigma_v\) denote the cross sectional area of the prismatic body and the silo pressure at the tunnel crown elevation, respectively:

\[F = B H \tan \omega,\]  

(7)

\[\sigma_v = \frac{R \gamma - c}{\lambda \tan \phi} \left(1 - e^{-\lambda \tan \phi \frac{T}{R}}\right)\]  

(8)

\[R = \frac{F}{U}\]  

(9)
where \( T \) = depth of cover; \( \lambda \) = coefficient of lateral stress = 0.8 – 1 and \( U \) = circumference of the cross section of the prismatic body:

\[
U = 2(B + H \tan \omega).
\] (10)

### 2.3 The shear force \( T_s \) acting at the lateral slip surfaces

The force \( T_s \) is calculated by integrating the respective shear stresses \( \tau \). The shear and normal stress at every point \((y, z)\) of the two lateral slip surfaces fulfill Coulomb criterion:

\[
\tau(y, z) = c + \sigma_x(y, z) \tan \phi,
\] (11)

where \( \sigma_x(y, z) \) denotes the stress normal to the lateral slip surface, i.e. in the horizontal direction. Analogously to silo theory, we assume that the ratio of horizontal stress \( \sigma_x \) to vertical stress \( \sigma_z \) is constant:

\[
\sigma_x(y, z) = \lambda_k \sigma_z(y, z),
\] (12)

where \( \lambda_k \) denotes the coefficient of lateral stress in the wedge (\( \lambda_k = 0.4-0.5 \)). Furthermore, the assumption is made that the vertical stress \( \sigma_z(y, z) \) in the wedge depends linearly on depth \( z \) according to Fig. 2-1a:

\[
\sigma_z(y, z) = (H - z) \gamma + \frac{z}{H} \sigma_{v,Keil},
\] (13)

where \( \sigma_{v,Keil} \) denotes the vertical stress at the upper boundary of the wedge (\( \sigma_{v,Keil} = \sigma_v \) according to Eq. 8). From Eqs. (11), (12) & (13) we obtain the shear stress as a function of the depth \( z \):

\[
\tau(z) = c + \lambda_k \tan \phi \left( (H - z) \gamma + \frac{z}{H} \sigma_v \right).
\] (14)

In order to obtain the resultant shear force \( T_s \), Eq. (14) has to be integrated over the height of the wedge:

\[
T_s = 2 \int_0^H \tau(z) b(z) dz,
\] (15)

where \( b(z) \) denotes the “width” of the wedge at elevation \( z \) (Fig. 5):

\[
b(z) = z \tan \omega.
\] (16)

From Eqs. (14), (15) & (16) we obtain:

\[
T_s = H^2 \tan \omega \left( c + \lambda_k \tan \phi \frac{2\sigma_v + H\gamma}{3} \right).
\] (17)
2.4 Special case: Short-term stability in a low permeability ground

Short-term stability is analysed based upon undrained shear strength $s_u$ ($\phi = 0$). Due to the absence of frictional resistance, the assumptions made concerning the horizontal stresses are not any more necessary and simpler expressions can be derived for the forces $T_s$ and $V$.

The silo pressure $\sigma_v$ results from the vertical equilibrium of the prismatic body by taking into account a constant shear stress $s_u$:

$$\sigma_v = T\gamma \left(1 - \frac{s_u}{R\gamma}\right). \quad (21)$$

Accordingly, the load acting upon the wedge is equal to the overburden pressure $T\gamma$ reduced by the factor $(1 - s_u/R\gamma)$.

Since the shear stress prevailing at the two lateral slip surfaces is also equal to the undrained shear strength $s_u$, the resultant shear force $T_s$ is given by:

$$T_s = s_u H^2 \tan \omega. \quad (22)$$

The necessary support force $S$ can be calculated from Eqs. (22), (5), (6), (18) and (4) with $\phi = 0$ and $c = s_u$:

$$S = BH \sigma_v + \frac{1}{2} \gamma BH^2 - s_u 2H \frac{H \sin \omega + B}{\sin 2\omega} \quad (23)$$

Note that for shear strengths $s_u$ higher than $R\gamma$, the silo pressure $\sigma_v$ becomes negative (see Eq. 21). In this case, the prismatic body is stable even without support and consequently it does not exert a force on the wedge. Eq. 23 should therefore be used with $\sigma_v = 0$. 
3 Summary of equations

General case

\[ S = \frac{F \alpha + G}{\tan(\omega + \phi)} - \frac{T_s + c B H}{\cos \omega (\tan \omega + \tan \phi)} \]  \quad (4)

\[ G = \frac{1}{2} B H^2 \tan \omega. \]  \quad (5)

\[ \alpha_v = \frac{R\gamma - c}{\lambda \tan \phi} \left( 1 - e^{-\lambda \tan \varphi \frac{F}{R}} \right). \]  \quad (8)

\[ T_s = H^2 \tan \omega \left( c + \lambda_k \tan \varphi \frac{2\alpha_v + H\gamma}{3} \right). \]  \quad (17)

Special case of short-term stability

\[ S = B H \alpha_v + \frac{1}{2} \gamma B H^2 - s_u B H \frac{H \sin \omega + B}{\sin 2\omega}. \]  \quad (23)

\[ \alpha_v = T \gamma \left( 1 - \frac{s_u}{R \gamma} \right). \]  \quad (21)

Geometrical parameters

\[ R = \frac{F}{U}, \quad F = B H \tan \omega, \quad U = 2(B + H \tan \omega). \]  \quad (9, 18, 19)

Remarks

- Consideration of a safety factor S.F.: Apply the equations given above with reduced shear strength constants, i.e. with \( (c/S.F., \tan \varphi/S.F., s_u/S.F.) \) instead of \( (c, \tan \varphi, s_u) \)!

- Negative values of silo pressure \( \alpha_v \) (Eqs. 8 & 21): Apply equations (4) and (23) \( \alpha_v = 0 \)!